

An Algorithm for Computing the Ratliff-Rush Closure

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Abstract

Let $I \subset K[x, y]$ be a $\langle x, y \rangle$ -primary monomial ideal where K is a field. This paper produces an algorithm for computing the Ratliff-Rush closure \tilde{I} for the ideal $I = \langle m_0, \dots, m_n \rangle$ whenever m_i is contained in the integral closure of the ideal $\langle x^{a_n}, y^{b_0} \rangle$. This generalizes of the work of Crispin [Cri]. Also, it provides generalizations and answers for some questions given in [HJLS], and enables us to construct infinite families of Ratliff-Rush ideals.

Let R be a commutative Noetherian ring with unity and I a regular ideal in R , that is, an ideal that contains a nonzerodivisor. Then the ideals of the form $I^{n+1} : I^n = \{x \in R \mid xI^n \subseteq I^{n+1}\}$ give the ascending chain $I : I^0 \subseteq I^2 : I^1 \subseteq \dots \subseteq I^n : I^{n+1} \subseteq \dots$. Let

$$\tilde{I} = \bigcup_{n \geq 1} (I^{n+1} : I^n).$$

As R is Noetherian, $\tilde{I} = I^{n+1} : I^n$ for all sufficiently large n . Ratliff and Rush [RR, Theorem 2.1] proved that \tilde{I} is the unique largest ideal for which $(\tilde{I})^n = I^n$ for sufficiently large n . The ideal \tilde{I} is called the *Ratliff-Rush closure* of I and I is called *Ratliff-Rush* if $I = \tilde{I}$.

As yet, there is no algorithm to compute the Ratliff-Rush closure for regular ideals in general. To compute $\bigcup_n (I^{n+1} : I^n)$ one needs to find a positive integer N such that $\bigcup_n (I^{n+1} : I^n) = I^{N+1} : I^N$. However, $I^{n+1} : I^n = I^{n+2} : I^{n+1}$ does not imply that $I^{n+1} : I^n = I^{n+3} : I^{n+2}$ ([RS], Example (1.8)). Several different approaches have been used to decide the Ratliff-Rush closure; Heinzer et al. [HLS], Property (1.2), established that every power of a regular ideal I is Ratliff-Rush if and only if the associated graded ring, $gr_I(R) = \bigoplus_{n \geq 0} I^n / I^{n+1}$, has a nonzerodivisor (has positive depth). Thus the Ratliff-Rush property of an ideal is a good tool for getting information about the depth of the graded associated ring which is a topic of interest for many authors such as [HM], [Hun] and [Ghe]. Al-Ayyoub [Ayy] used a technique that depends on the degree count to prove that certain monomial ideals (that are the defining ideal of certain monomial curves) are Ratliff-Rush, namely, if the ideal $I \subseteq K[x_1, \dots, x_n]$ with K a field is primary to (x_1, \dots, x_n) and $\tilde{I} \cap (I : (x_1, \dots, x_n)) \subseteq I$, then I is Ratliff-Rush (for a proof see either Theorem (1.3) in [Ayy] or Proposition (15.4.1) in [SH]). Elias [Elias] established a procedure for computing the Ratliff-Rush closure of \mathbf{m} -primary ideals of a Cohen-Macaulay local ring with maximal ideal \mathbf{m} . Elias' procedure depends on computing the Hilbert-Poincaré series of I and then the multiplicity and the postulation number of I .

Let $I \subset K[x, y]$ be a $\langle x, y \rangle$ -primary monomial ideal with $I = \langle m_0, \dots, m_n \rangle$ where $m_i = x^{a_i} y^{b_i}$ for $i = 0, \dots, n$ with $a_0 = b_n = 0$. That is, $I = \langle y^{b_0}, x^{a_1} y^{b_1}, \dots, x^{a_{n-1}} y^{b_{n-1}}, x^{a_n} \rangle$. In

this paper we produce an algorithm for computing the Ratliff-Rush closure \tilde{I} for the ideal I whenever $m_i \in I(a_n, b_0)$, the integral closure of the ideal $\langle x^{a_n}, y^{b_0} \rangle$ (see the definition of the integral closure in the beginning of the next section). This gives a generalization of the work of Crispin [Cri]. This algorithm provides generalizations and answers for some questions given in [HJLS]. Also, it enables us to construct infinite families of Ratliff-Rush ideals. We may say that the algorithm we provide in this paper is the very first explicit algorithm, for computing the Ratliff-Rush closure for a wide range of monomial ideals in polynomial rings with two indeterminates, as no theoretical background is needed, that is, the algorithm depends only on elementary computations on numerical semigroups.

The algorithm is simple enough to be introduced right away and demonstrated on an example: let Ω be the numerical semigroup in \mathbb{Z}^2 generated by the set $\{(a_i, b_i) \mid i = 0, \dots, n\}$, that is, $\Omega = \{(\alpha, \beta) = \sum_{i=0}^n \lambda_i (a_i, b_i) \mid \lambda_i \in \mathbb{Z}_{\geq 0}\}$. Let

$$S = \{(\alpha, \beta) \mid \alpha \leq a_n, \beta \leq b_0, \text{ and } (\alpha, \beta + kb_0) \in \Omega \text{ for some } k \in \mathbb{Z}_{\geq 0}\},$$

and

$$T = \{(\alpha, \beta) \mid \alpha \leq a_n, \beta \leq b_0, \text{ and } (\alpha + ka_n, \beta) \in \Omega \text{ for some } k \in \mathbb{Z}_{\geq 0}\}.$$

Set

$$I_S = \langle x^\alpha y^\beta \mid (\alpha, \beta) \in S \rangle \text{ and } I_T = \langle x^\alpha y^\beta \mid (\alpha, \beta) \in T \rangle.$$

Then we show that $\tilde{I} = I_S \cap I_T$.

Before proceeding to prove this result we would like to demonstrate it by the example below. The reader may have a look at Example (13) which might give an easier representation. A semigroup S in \mathbb{Z}^2 is said to be minimally generated by a set $A \subseteq S$ if A is the smallest subset in S such that whenever $(\alpha, \beta) \in S$, then there exists $(\alpha', \beta') \in A$ such that $\alpha' \leq \alpha$ and $\beta' \leq \beta$.

Example 1 Let $I = \langle y^{28}, x^2 y^{26}, x^{10} y^{14}, x^{11} y^{12}, x^{15} y^5, x^{17} \rangle \subseteq I(17, 28) \subset K[x, y]$. Then $\Omega = \langle p_0, p_1, p_2, p_3, p_4, p_5 \rangle$ where $p_0 = (a_0, b_0) = (0, 28)$, $p_1 = (a_1, b_1) = (2, 26)$, $p_2 = (a_2, b_2) = (10, 14)$, $p_3 = (a_3, b_3) = (11, 12)$, $p_4 = (a_4, b_4) = (15, 5)$, and $p_5 = (a_5, b_5) = (17, 0)$. To compute S consider $\{(\alpha, \beta) \in \Omega \text{ and } \alpha \leq 17\} = \{\sum_{i=0}^5 \lambda_i p_i \mid \lambda_0 \in \mathbb{Z}_{\geq 0}, \lambda_1 \leq 8, \lambda_i \leq 1 \text{ for } 2 \leq i \leq 5\}$. Now S is minimally generated by $\{p_0, p_1, p_2, p_3, p_4, p_5\} \cup \{(4, 24), (6, 22), (8, 20), (13, 10)\}$ as $(4, 24) = (2a_1, 2b_1 \bmod 28)$, $(6, 22) = (3a_1, 3b_1 \bmod 28)$, $(8, 20) = (4a_1, 4b_1 \bmod 28)$, and $(13, 10) = (a_1 + a_3, b_1 + b_3 \bmod 28)$. Thus

$$I_S = \langle y^{28}, x^2 y^{26}, x^4 y^{24}, x^6 y^{22}, x^8 y^{20}, x^{10} y^{14}, x^{11} y^{12}, x^{13} y^{10}, x^{15} y^5, x^{17} \rangle.$$

Similarly, T is minimally generated by $\{p_0, p_1, p_2, p_3, p_4, p_5\} \cup \{(13, 10), (9, 17), (8, 19), (7, 22), (5, 24)\}$ as $(13, 10) = (2a_4 \bmod 17, 2b_4)$, $(9, 17) = (a_3 + a_4 \bmod 17, b_3 + b_4)$, $(8, 19) = (a_2 + a_4 \bmod 17, b_2 + b_4)$, $(7, 22) = (a_3 + 2a_4 \bmod 17, b_3 + 2b_4)$, and $(5, 24) = (2a_3 \bmod 17, 2b_3)$. Thus

$$I_T = \langle y^{28}, x^2 y^{26}, x^5 y^{24}, x^7 y^{22}, x^8 y^{19}, x^9 y^{17}, x^{10} y^{14}, x^{11} y^{12}, x^{13} y^{10}, x^{15} y^5, x^{17} \rangle.$$

Therefore, $\tilde{I} = I_S \cap I_T = \langle y^{28}, x^2 y^{26}, x^5 y^{24}, x^7 y^{22}, x^8 y^{20}, x^{10} y^{14}, x^{11} y^{12}, x^{13} y^{10}, x^{15} y^5, x^{17} \rangle$.

The author would like to point out that the algorithm that is provided in this paper does not apply to arbitrary monomial ideals in $K[x, y]$ as it will be illustrated at the end of the next section.

1 Decomposition of powers of an ideal

We start by decomposing sufficiently large powers of the ideal I by means of the semigroups S and T , see Lemma (6) below. In order to do so we need to consider some remarks concerning the semigroups S and T and the hypothesis $m_i \in I(a_n, b_0)$, the integral closure of the ideal $\langle x^{a_n}, y^{b_0} \rangle$ as we define now:

Definition 2 *Let I be an ideal in a Noetherian ring R . The integral closure of I is the ideal \bar{I} that consists of all elements of R that satisfy an equation of the form*

$$x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = 0, \quad a_i \in I^i.$$

The ideal I is said to be integrally closed if $I = \bar{I}$.

It is well known that the integral closure of monomial ideal in a polynomial ring is again a monomial ideal (See [SH], Proposition 1.4.2). The problem of finding the integral closure for a monomial ideal I reduces to finding monomials r , integer i and monomials m_1, m_2, \dots, m_i in I such that $r^i = m_1 m_2 \cdots m_i$, see Section 1.4 in [SH]. Geometrically, finding the integral closure of monomial ideals I in $R = K[x_0, \dots, x_n]$ is the same as finding all the integer lattice points in the convex hull $NP(I)$ (the Newton polyhedron of I) in \mathbb{R}^n of $\Gamma(I)$ (the Newton polytope of I) where $\Gamma(I)$ is the set of all exponent vectors of all the monomials in I . This implies $x_1^{\gamma_1} x_2^{\gamma_2} \cdots x_n^{\gamma_n} \in I(a_1, a_2, \dots, a_n)$ if and only if there are non-negative rational numbers c_1, c_2, \dots, c_n with $\sum_{i=1}^n c_i = 1$ and $\gamma_i \geq c_i a_i$.

Remark 3 *Let $I = \langle m_0, \dots, m_n \rangle$ where $m_i = x^{a_i} y^{b_i} \in I(a_n, b_0)$ for $i = 0, \dots, n$ with $a_0 = b_n = 0$.*

- (1) $I^l = \left\langle x^\alpha y^\beta : (\alpha, \beta) = \sum_{i=0}^n \lambda_i (a_i, b_i) \in \Omega \text{ and } \sum_{i=0}^n \lambda_i = l \right\rangle$ for all $l \in \mathbb{Z}^+$.
- (2) If $x^\alpha y^\beta \in I^l$, then $\frac{\alpha}{a_n} + \frac{\beta}{b_0} \geq l$. In particular, either $\beta \geq (l/2) b_0$ or $\alpha \geq (l/2) a_n$.
- (3) $\frac{b_0 - b_i}{a_i} \leq \frac{b_0}{a_n}$ for $i = 1, \dots, n-1$.

Proof. As $m_i = x^{a_i} y^{b_i} \in I(a_n, b_0)$, then there exist $c_1, c_2 \in \mathbb{Q}^+$ with $c_1 + c_2 = 1$ such that $a_i \geq c_1 a_n$ and $b_i \geq c_2 b_0$. Hence, $\frac{a_i}{a_n} + \frac{b_i}{b_0} \geq 1$, this implies $\frac{\alpha'}{a_n} + \frac{\beta'}{b_0} = \sum_{i=0}^n \lambda_i \left(\frac{a_i}{a_n} + \frac{b_i}{b_0} \right) \geq \sum_{i=0}^n \lambda_i = l$. Also, $\frac{b_0 - b_i}{a_i} \leq \frac{b_0 - b_i}{c_1 a_n} \leq \frac{b_0 - c_2 b_0}{c_1 a_n} = \frac{b_0}{a_n} \frac{1 - c_2}{c_1} = \frac{b_0}{a_n}$. ■

The following remark provides us with the technique that we repeatedly use in this paper

Remark 4 *If $(\alpha, \beta) = \sum_{i=0}^n \lambda_i (a_i, b_i) \in \Omega$ with $\sum_{i=0}^n \lambda_i = l$ and $\alpha = \sum_{i=0}^n \lambda_i a_i \leq a_n$, then $\beta = \sum_{i=0}^n \lambda_i b_i = (l-1) b_0 + \beta_1$ with $(\alpha, \beta_1) \in S$. Also, if $(\alpha, \beta) \in S$ with $(\alpha, \beta + k b_0) \in \Omega$ and $\alpha = \sum_{i=0}^n \lambda_i a_i$, $\beta + k b_0 = \sum_{i=0}^n \lambda_i b_i$ with $\sum_{i=0}^n \lambda_i = l$, then $k = l-1$.*

Proof. Showing $\beta = \sum_{i=0}^n \lambda_i b_i = (l-1) b_0 + \beta_1$ with $\beta_1 \leq b_0$ is equivalent to showing that $\sum_{i=0}^n \lambda_i (b_0 - b_i) \leq b_0$. Since $\alpha = \sum_{i=0}^n \lambda_i a_i \leq a_n$, then $\sum_{i=0}^n \lambda_i \frac{a_i}{a_n} \leq 1$. Thus by part (3) of Remark (3) we get $\sum_{i=0}^n \lambda_i (b_0 - b_i) \leq \sum_{i=0}^n \lambda_i \frac{a_i}{a_n} b_0 \leq b_0$.

To prove the other part it is enough to show $\beta + kb_0 \geq (l-1)b_0$. Consider $(l-1)b_0 - (\beta + kb_0) = lb_0 - \sum_{i=0}^n \lambda_i b_i - b_0 = \sum_{i=0}^n \lambda_i (b_0 - b_i) - b_0 \leq 0$ since $\sum_{i=0}^n \lambda_i (b_0 - b_i) \leq b_0$ as above. ■

Notation 5 Let $q_S, q_T \in \mathbb{Z}$ be such that $a_n = q_S a_1 + e_S$ with $0 \leq e_S < a_1$ and $b_n = q_T b_{n-1} + e_T$ with $0 \leq e_T < b_{n-1}$. Note that if $(\alpha, \beta) \in S$ with $\alpha = \sum_{i=0}^n \lambda_i a_i$, then $\sum_{i=0}^n \lambda_i \leq q_S$. And if $(\alpha, \beta) \in T$ with $\beta = \sum_{i=0}^n \delta_i b_i$, then $\sum_{i=0}^n \delta_i \leq q_T$. Also, note $q_S, q_T \geq 1$.

This section is concluded with an explicit decomposition of sufficiently large powers of the ideal I . This decomposition enables us to compute the Ratliff-Rush closure.

Lemma 6 Let I , I_S , and I_T be as above. Then for every $l \geq \max\{q_T, q_S\}$

$$I^l = y^{b_0(l-1)} I_S + x^{a_n(l-1)} I_T + x^{a_n} y^{b_0} M$$

where $M = I^l : (x^{a_n} y^{b_0})$.

Proof. If $x^\alpha y^\beta \in I_T$, then $\beta = \sum_{i=0}^n \lambda_i b_i \leq b_0$ and $\alpha = \sum_{i=0}^n \lambda_i a_i - ca_n$ for some positive integer c with $q_T \geq \sum_{i=0}^n \lambda_i > c$. Now if $l \geq q_T$, then $x^{a_n(l-1)} x^\alpha y^\beta = (x^{a_n})^{l-(c+1)} \prod_{i=0}^n (x^{a_i} y^{b_i})^{\lambda_i} \in I^l$ as $l - (c+1) + \sum_{i=0}^n \lambda_i \geq l$. Similarly, if $l \geq q_S$, then $y^{b_0(l-1)} x^\alpha y^\beta \in I^l$ for every $x^\alpha y^\beta \in I_S$.

For the other inclusion it is enough to show that if $x^\alpha y^\gamma \in I^l$ with $\alpha \leq a_n$, then $x^\alpha y^\gamma \in y^{b_0(l-1)} I_S$. But this is done by part (1) of Remark (3) and Remark (4) as $\alpha = \sum_{i=0}^n \lambda_i a_i \leq a_n$ and $\gamma = \sum_{i=0}^n \lambda_i b_i = (l-1)b_0 + \beta$ with $(\alpha, \beta) \in I_S$. ■

Considering the ideal $I = \langle x^7, x^6 y, xy^{10}, y^{14} \rangle$, the reader can easily see that any power of I does not satisfy the above decomposition which is a cornerstone of the main result of this paper. This causes the algorithm not to be applicable for arbitrary monomial ideals.

2 Powers of an ideal and the Ratliff-Rush closure

In the lemma below we show that the generators of a sufficiently large power of I take a patterns that involve powers of m_0 and m_n . This is a consequences of the hypothesis on the generators of I , that is, $m_i = x^{a_i} y^{b_i} \in I(a_n, b_0)$.

Lemma 7 Let $I = \langle m_n, \dots, m_0 \rangle$ where $m_i = x^{a_i} y^{b_i} \in I(a_n, b_0)$ for $i = 0, \dots, n$ and $a_0 = b_n = 0$. Let r be any positive integer. Then there exist a positive integer L such that if $l \geq L$, then the generators of I^l are of the forms $m_0^\gamma \xi_{0,\gamma} m_n^{l-\gamma-1}$ and $m_0^{l-\gamma-1} \xi_{\gamma,0} m_n^\gamma$ for every γ with $r \leq \gamma \leq l-1$ where $\xi_{0,\gamma}$ and $\xi_{\gamma,0}$ are some monomials.

Proof. Let r be a positive integer and $q = \max\{q_T, q_S\}$. Choose $L = 2(r+1)$ and let $\omega = x^{\alpha'} y^{\beta'} \in I^l$ for some $l \geq L$. By part (1) of Remark (3) we may write $(\alpha', \beta') =$

$\sum_{i=0}^n \lambda_i (a_i, b_i) \in \Omega$ with $\sum_{i=0}^n \lambda_i = l$. By part (2) of Remark (3) either $\beta' \geq (r+1)b_0$ or $\alpha' \geq (r+1)a_n$. We make the proof whenever $\beta' \geq (r+1)b_0$ where we show $\omega = m_0^\gamma \xi_{0,\gamma} m_n^{l-\gamma-1}$ for some monomial $\xi_{0,\gamma}$. The proof is similar for the case $\alpha' \geq (r+1)a_n$ where it can be shown that $\omega = m_0^{l-\gamma-1} \xi_{\gamma,0} m_n^\gamma$.

Let $\beta' \geq (r+1)b_0$ and write $\beta' = \gamma b_0 + \beta$ with $0 \leq \beta < b_0$. Note $r < \gamma < l$ as $(r+1)b_0 \leq \beta' \leq l b_0$. Since $\frac{\beta'}{b_0} < \gamma + 1$, then by part (2) of Remark (3) we must have $\alpha' \geq (l - \gamma - 1)a_n$. Write $\alpha' = (l - \gamma - 1)a_n + \alpha$. Now $\omega = x^{\alpha'} y^{\beta'} = (y^{b_0})^\gamma x^\alpha y^\beta (x^{a_n})^{l-\gamma-1} = m_0^\gamma x^\alpha y^\beta m_n^{l-\gamma-1}$.

Finally, note $\langle x^{a_n}, y^{b_0} \rangle \subseteq I^l$, hence $\langle y^{lb_0}, y^{(l-1)b_0} x^{a_n}, \dots, y^{(r+1)b_0} x^{l-(r+1)a_n}, \dots, x^{a_n} \rangle \subseteq I^l$ which suffices to show that γ takes all integer values between r and $l-1$, which finishes the proof. ■

Remark 8 If $r = 2q$ and $l \geq 4q + 2$ as in the above lemma and $\omega \in I^l$, then $\omega \in (m_0^q)I^{l-q}$ or $\omega \in I^{l-q}(m_n^q)$.

Proof. Assume $\omega = m_0^\gamma \xi_{0,\gamma} m_n^{l-\gamma-1} \in I^l$ with $r \leq \gamma \leq l-1$. Applying the above lemma with $r' = q-1$ and $l' \geq 2q$, then the generators of $I^{l'}$ are of the forms $m_0^{\gamma'} \xi_{0,\gamma'} m_n^{l-\gamma'-1}$ and $m_0^{l-\gamma'-1} \xi_{\gamma',0} m_n^{\gamma'}$ for every γ' with $q-1 \leq \gamma' \leq l-1$ where $\xi_{0,\gamma'}$ and $\xi_{\gamma',0}$ are some monomials. As $2q \leq \gamma$ and $l \geq 4q + 2$, then $r' \leq \gamma - q - 1 \leq l - q - 2$ and $l - q - 2 \geq 2q$. Thus setting $\gamma' = \gamma - q - 1$ and $l' = l - q - 2$. Therefore, $m_0^{\gamma-q-1} \xi_{0,\gamma'} m_n^{l-\gamma-2} = m_0^{\gamma'} \xi_{0,\gamma'} m_n^{l-\gamma'-1} \in I^{l-q}$. Note $m_0 \xi_{0,\gamma} m_n$ is a multiple of $\xi_{0,\gamma'}$, say $m_0 \xi_{0,\gamma} m_n = \rho \xi_{0,\gamma'}$. Thus $\omega = m_0^\gamma \xi_{0,\gamma} m_n^{l-\gamma-1} = m_0^{\gamma-1} (m_0 \xi_{0,\gamma} m_n) m_n^{l-\gamma-2} = m_0^q \rho (m_0^{\gamma-q-1} \xi_{0,\gamma'} m_n^{l-\gamma-2}) \in (m_0^q)I^{l-q}$.

Assume $\omega = m_0^{l-\gamma-1} \xi_{\gamma,0} m_n^\gamma$. Then a similar process shows that $\omega \in I^{l-q}(m_n^q)$, which finishes the proof. ■

Now we are ready to prove the first main theorem of the paper.

Theorem 9 Let the ideals I , I_S , and I_T be as before. Then $\tilde{I} = I_S \cap I_T$.

Proof. Let $\delta \in I_S \cap I_T$ and $q = \max\{q_S, q_T\}$. Claim $\delta m_0^q, \delta m_n^q \in I^{q+1}$: as $\delta \in I_S$, then $\delta = x^r y^s$ with $(r-u, s-v) \in S$ for some positive integers u and v , that is, $r-u = \sum_{i=0}^n \lambda_i a_i \leq a_n$ and $s-v \leq b_0$ with $(r-u, s-v+kb_0) \in \Omega$ and $s-v+kb_0 = \sum_{i=0}^n \lambda_i b_i$. Let $t = \sum_{i=0}^n \lambda_i$. By Notations (5) we have $t \leq q$, and by Remark (4) we may rewrite $\sum_{i=0}^n \lambda_i b_i = (t-1)b_0 + s-v$. Then

$$\begin{aligned} \delta m_0^q &= x^r y^s (y^{b_0})^q = x^u y^v x^{r-u} y^{s-v} (y^{b_0})^{t-1} (y^{b_0})^{q-(t-1)} \\ &= x^u y^v x^{\left(\sum_{i=0}^n \lambda_i a_i\right)} y^{\left(\sum_{i=0}^n \lambda_i b_i\right)} (y^{b_0})^{q-(t-1)} \\ &= x^u y^v \prod_{i=0}^n (x^{a_i} y^{b_i})^{\lambda_i} (y^{b_0})^{q-(t-1)} \in I^{q+1} \end{aligned}$$

Similarly, as $\delta \in I_T$, then by a similar procedure as above it can be shown that $\delta m_n^q \in I^{q+1}$.

Now choose $r = 2q$, then by Lemma (7) if $l \geq 2(2q + 1)$, then any generator ω of I^l is either of the form $m_0^\gamma \xi_{0,\gamma} m_n^{l-\gamma-1}$ or $m_0^{l-\gamma-1} \xi_{\gamma,0} m_n^\gamma$ for every γ with $r \leq \gamma \leq l-1$ where $\xi_{0,\gamma}$ and $\xi_{\gamma,0}$ are some monomials.

Assume $\omega = m_0^\gamma \xi_{0,\gamma} m_n^{l-\gamma-1}$. By Remark (8) we have $\omega \in (m_0^q) I^{l-q}$. Therefore, by the claim above $\delta\omega \in (\delta m_0^q) I^{l-q} \in I^{q+1} I^{l-q} = I^{l+1}$.

Assume $\omega = m_0^{l-\gamma-1} \xi_{\gamma,0} m_n^\gamma$. By Remark (8) we have $\omega \in I^{l-q} (m_n^q)$. Therefore, by the claim above $\delta\omega \in I^{l-q} (\delta m_n^q) \in I^{l-q} I^{q+1} = I^{l+1}$.

On the other hand, assume $\delta \notin I_S$ and let l be any positive integer. Then $\delta y^{lb_0} \notin y^{lb_0} I_S$, also $\delta y^{lb_0} \notin x^{la_n} I_T$ and $\delta y^{lb_0} \notin (x^{a_n} y^{b_0}) M$ because of the y -degree count where $M = I^l : (x^{a_n} y^{b_0})$. Hence, $\delta y^{lb_0} \notin I^{l+1}$ by Lemma (6). Analogously, if $\delta \notin I_T$, then $\delta x^{la_n} \notin I^{l+1}$, which finishes the proof. ■

Remark 10 *The Ratliff-Rush reduction number of an ideal I is defined $r(I) = \min\{l \in \mathbb{Z}_{\geq 0} \mid \tilde{I} = (I^{l+1} : I^l)\}$. From the proof of Theorem (9) it is clear that $2q$ is an upper bound for the Ratliff-Rush reduction number of the ideal I .*

3 Consequences and Examples

Heinzer et al. [HJLS], Example (6.3), conjectured that for any integer d the ideal $I_d = \langle x^d, x^{d-1}y, y^d \rangle$ and all its powers are Ratliff-Rush. This conjectured was proved later by [RS], Proposition (1.9), by actual computations of the depth of gr_{I_d} , the associated graded ring of I_d . Later [Cri], Example (4.2), proved this conjecture by a method that we generalize in the paper. In Corollary (12) below we give a generalization of this conjecture.

Remark 11 *Let $I = \langle y^{b_0}, x^{a_1} y^{b_1}, \dots, x^{a_{n-1}} y^{b_{n-1}}, x^{a_n} \rangle$ with $m_i \in I(a_n, b_0)$. Then I is Ratliff-Rush if any of the following holds.*

- (1) $a_i \geq a_n/2$ for all i or $b_i \geq b_0/2$ for all i .
- (2) For all i and j either $a_i + a_j \geq a_n$ or $a_i + a_j = a_k$ and $(b_i + b_j) \bmod b_0 \geq b_k$ for some k .

Powers of a Ratliff-Rush ideal need not be Ratliff-Rush as Example (6.1) of [HJLS] shows. As the powers of an ideal are Ratliff-Rush implies that the associated graded ring, $gr_I(R) = \bigoplus_{n \geq 0} I^n / I^{n+1}$, has a positive depth, we will investigate the Ratliff-Rush closedness for all powers of ideals in the remaining of the paper.

Corollary 12 *Let $I = \langle x^c, m, y^d \rangle$ where $m = 0$ or $m = x^u y^v \in I(c, d)$. Then all powers of I are Ratliff-Rush.*

Proof. if $m = 0$, then $I = \langle x^c, y^d \rangle$. Thus $S = T = \{(0, c), (d, 0)\}$, hence $I_S = I = I_T$. Also, $I^l = \langle y^{ld}, x^c y^{(l-1)d}, x^{2c} y^{(l-2)d}, \dots, x^{(l-2)c} y^{2d}, x^{(l-1)c} y^d, x^{lc} \rangle$. It is clear that $(I^l)_S = I^l$ and $(I^l)_T = I^l$. Thus $\tilde{I}^l = I^l$.

Assume $m = x^u y^v$, then by part (2) of Remark (3) either $u \geq c/2$ or $v \geq d/2$. Thus $\tilde{I} = I$ by part (1) of Remark (11). Let $J = I^l$. Consider

$$\begin{aligned} J &= \langle y^{ld}, x^u y^{(l-1)d+v}, x^{2u} y^{(l-2)d+2v}, \dots, x^{lu} y^{lv} \rangle + \\ &\quad \langle x^{lu} y^{lv}, x^{(l-1)u+c} y^{(l-1)v}, x^{(l-2)u+2c} y^{(l-2)v}, \dots, x^{2u+(l-2)c} y^{2v}, x^{u+(l-1)c} y^v, x^{lc} \rangle \\ &= \langle x^{iu} y^{(l-i)d+iv} \mid i = 0, \dots, l \rangle + \langle x^{iu+(l-i)c} y^{iv} \mid i = 0, \dots, l \rangle. \end{aligned}$$

Let

$$K = \{(a_i, b_i) = (iu, (l-i)d + iv) \mid i = 0, \dots, l\}$$

and

$$H = \{(a_{2l-i+1}, b_{2l-i+1}) = (iu + (l-i)c, iv) \mid i = 0, \dots, l\}.$$

Then $J = \langle x^a y^b \mid (a, b) \in K \cup H \rangle = \langle x^{a_i} y^{b_i} \mid i = 0, \dots, 2l+1 \rangle$. See Figure (1) for a representation of J . Note that if $(a_i, b_i) \in K$, then $b_i \geq lv$, and if $(a_i, b_i) \in H$, then $a_i \geq lu$.

Claim: if $(\alpha, \beta) \in S \setminus (K \cup H)$, then $\alpha \geq lu$, and if $(\alpha', \beta') \in T \setminus (K \cup H)$, then $\beta' \geq lv$. We prove the first part of the claim and the second part is similar. Assume $(\alpha, \beta) \in S$ with $\alpha < lu$. As $\alpha = \sum_{i=0}^{2l+1} \lambda_i a_i < lu$, then $\lambda_i = 0$ whenever $(a_i, b_i) \in H$. Hence we must have

$\alpha = \sum_{i=0}^l \lambda_i a_i$, that is $(a_i, b_i) \in K$, or $(a_i, b_i) = (iu, (l-i)d + iv)$. Thus $\alpha = \sum_{i=0}^l (i\lambda_i)u$ and

$$\begin{aligned} \beta &= \left(\sum_{i=0}^l \lambda_i (l-i)d + \sum_{i=0}^l (i\lambda_i)v \right) \bmod ld \\ &= \left(\sum_{i=0}^l (\lambda_i - 1)ld + \sum_{i=0}^l [(l-i\lambda_i)d + (i\lambda_i)v] \right) \bmod ld \\ &= \left(\sum_{i=0}^{l-1} ld + (l - \sum_{i=0}^l i\lambda_i)d + \left(\sum_{i=0}^l i\lambda_i \right)v \right) \bmod ld \\ &= (l - \sum_{i=0}^l i\lambda_i)d + \left(\sum_{i=0}^l i\lambda_i \right)v. \end{aligned}$$

Now as $\alpha = \sum_{i=0}^l (i\lambda_i)u < lu$, then $\sum_{i=0}^l (i\lambda_i) < l$ and hence $(\alpha, \beta) \in K$.

Now by the claim, if $x^\alpha y^\beta \in J_S \cap J_T$, then either $x^\alpha y^\beta = x^{a_i} y^{b_i} \in J$ for some $i = 0, \dots, 2l+1$, or $x^\alpha y^\beta \in \langle x^{lu} y^{lv} \rangle \subseteq J$. ■

Figure 1. A representation of I^5 where $I = \langle x^7, x^5 y^2, y^5 \rangle$. The circles (black or white) represent the set K , and the black points (circles or squares) represent the set H . The white square represents a monomial in I_T but not in I .

The above corollary showed that all powers of a (x, y) -primary monomial ideal with three generators, satisfying the underlined conditions, are Ratliff-Rush. This is not the case if the ideal is generated by 4 elements as the example below shows.

Example 13 Let $I = \langle x^{35}, x^{33}y^2, x^4y^{26}, y^{28} \rangle$. Then $S = \{(0, 28), (4, 26), (8, 24), (12, 22), (16, 20), (20, 18), (24, 16), (28, 14), (32, 12), (33, 2), (35, 0)\}$ and $T = \{(35, 0), (33, 2), (31, 4), (29, 6), (27, 8), (25, 10), (23, 12), (21, 14), (19, 16), (17, 18), (15, 20), (13, 22), (11, 24), (4, 26), (0, 28)\}$ (see the figure below for illustration). Thus $\tilde{I} = \langle x^{35}, x^{33}y^2, x^{32}y^{12}, x^{28}y^{14}, x^{24}y^{16}, x^{20}y^{18}, x^{16}y^{20}, x^{13}y^{22}, x^{11}y^{24}, x^4y^{26}, y^{28} \rangle$.

Figure 2. A representation of $I_S \setminus I$ (white circles) and $I_T \setminus I$ (black squares) where $I = \langle x^{35}, x^{33}y^2, x^4y^{26}, y^{28} \rangle$. The points with a slash mark represent the monomials in $\tilde{I} \setminus I$.

Crispin [Cri] showed that for any d and k the ideal $I_{d,k} = \langle y^d, x^{d-k}y^k, x^{d-k+1}y^{k-1}, \dots, x^{d-1}y, x^d \rangle$ and all its powers are Ratliff-Rush. Also in Example (4.4) she showed that the ideal $I_{m,k} = \langle x^{im}y^{m(k+1-i)-1} \rangle_{i=0}^k + \langle x^{km+j}y^{m-j-1} \rangle_{j=0}^{m-1}$ and all its powers are Ratliff-Rush. In corollary (15) below we generalize this. First consider the notations below and the figures for illustration of the hypothesis of the corollary.

Notation 14 Let $c \leq d$ be two integers and $\mu_i = \lceil (c-i) \frac{d}{c} \rceil$. Let $c = n_1c_1 + n_2c_2 + \dots + n_rc_r$ with c_{i+1} divides c_i and $n_i \in \mathbb{Z}^+$ for all i and let $n_0 = 1$ and $c_0 = 0$. Also let $\sigma_{j,q} = qc_{j+1} + \sum_{i=1}^j n_i c_i$. Note the following

- (1) $\sigma_{-1,1} = 0$, hence $\mu_{\sigma_{-1,1}} = d$.
- (2) $\sigma_{0,q} = qc_1$ for $q = 1, \dots, n_1$.
- (3) $\sigma_{r-1,n_r} = c$.

Define the ideal $I = \langle x^{\sigma_{j,q}} y^{\mu_{\sigma_{j,q}}} \mid j = -1, 0, \dots, r-1 \text{ and } q = 1, 2, \dots, n_{j+1} \rangle$.

Note that if $c = d$ and if we choose $r = 2, c_1 = k, n_1 = 1, c_2 = 1$, and $n_2 = d - k$, then we get the ideal $I = I_{d,k}$ mentioned above. Also, if m and k are integers and if $c = d = m(k + 1) - 1$ and if we choose $r = 2, c_1 = m, n_1 = k, c_2 = 1$, and $n_2 = m - 1$, then we get the ideal $I = I_{m,k}$ as above.

Figure 3. $d = 20, c = 17, r = 3, c_1 = 4,$
 $n_1 = 2, c_2 = 2, n_2 = 3, c_3 = 1, n_3 = 3.$

Figure 4. $d = 20, c = 17, r = 2,$
 $c_1 = 5, n_1 = 1, c_2 = 1, n_2 = 12.$

Corollary 15 *All powers of the ideal $I = \langle x^{\sigma_{j,q}} y^{\mu_{\sigma_{j,q}}} \mid j = -1, 0, \dots, r-1 \text{ and } q = 1, 2, \dots, n_{j+1} \rangle$ are Ratliff-Rush.*

Proof. First note that since c_{i+1} divides c_i , then $\sigma_{j,q}$ can be written as tc_{j+1} for some $t < n_{j+1}$, or as $n_{j+1}c_{j+1} + n_{j+2}c_{j+2} + \dots + n_{j+e}c_{j+e} + tc_{j+e+1}$ for some e and $t < n_{j+e+1}$. Thus if $\sigma_{j_1,q_1} + \sigma_{j_2,q_2} \leq c$, then $\sigma_{j_1,q_1} + \sigma_{j_2,q_2} = \sigma_{j,q}$ for some $j \geq \max\{j_1, j_2\}$. Also $2d > \mu_{\sigma_{j_1,q_1}} + \mu_{\sigma_{j_2,q_2}} \geq d + \lceil (c - \sigma_{j,q}) \frac{d}{c} \rceil$, hence $(\mu_{\sigma_{j_1,q_1}} + \mu_{\sigma_{j_2,q_2}}) \bmod d \geq \lceil (c - \sigma_{j,q}) \frac{d}{c} \rceil = \mu_{\sigma_{j,q}}$. Thus I is Ratliff-Rush by part (2) of Remark (11).

Let $\omega = x^a y^b$ and $\omega' = x^{a'} y^{b'}$ be generators of I^l for some l . If $a + a' \leq lc$, then by the above paragraph we may write $a + a' = \sum_{j=-1}^{r-1} \sum_{q=1}^{n_{j+1}} \lambda_{j,q} \sigma_{j,q}$ with $\sum_{j=-1}^{r-1} \sum_{q=1}^{n_{j+1}} \lambda_{j,q} = l$ and also $b + b' \geq ld + \lceil (lc - a + a') \frac{d}{c} \rceil$. Hence, $(b + b') \bmod ld \geq \lceil (lc - (a + a')) \frac{d}{c} \rceil = \mu_{a+a'}$. Thus I^l is Ratliff-Rush by part (2) of Remark (11). ■

Remark 16 *For $c \leq d$, it is known that $I(c, d) = \langle x^i y^{\mu_i} \mid \mu_i = \lceil (c - i) \frac{d}{c} \rceil, i = 0, \dots, c \rangle$, the integral closure of the ideal $\langle x^c, y^d \rangle$.*

Heinzer et al. [HJLS] asked, Question (1.6) (Q1), whether the minimal number of generators of a regular ideal is always less than or equal to the minimal number of generators of its Ratliff-Rush closure. Rossi and Swanson [RS] answer this question in the following example. In the corollary below we answers this question by constructing an infinite family of monomial ideals I , with fewer variables, such that the minimal number of generators of I , $\mu(I)$, is arbitrary large and the minimal number of generators of \tilde{I} is 5 (see the figure below for illustration):

Example 17 ([RS], Example 3.6) Let F be a field, $n \geq 2$ an integer, x, y, z_1, \dots, z_n variables over F , $R = F[x, y, z_1, \dots, z_n]$, and $I = \langle x^4, x^3y, xy^3, y^4 \rangle + (x^2y^2) \langle z_1, \dots, z_n \rangle$. Then $\tilde{I} = (x, y)^4$, the minimal number of generators of I is $4 + n$ and the minimal number of generators of \tilde{I} is 5.

Corollary 18 Let $c \leq d$ be two integers each of which is divisible by 4 and $\mu_i = \lceil (c - i) \frac{d}{c} \rceil$. Let $I = \langle y^d, x^{c/4}y^{\mu_{c/4}}, x^{3c/4}y^{\mu_{3c/4}}, x^c \rangle + J$ where $J = \langle x^{c/2+c/4-1}y^{\mu_{c/2}}, x^{c/2+c/4-2}y^{\mu_{c/2}+2}, \dots, x^{c/2}y^{\mu_{c/2}+c/4+1} \rangle$. Then $\tilde{I} = \langle y^d, x^{c/4}y^{\mu_{c/4}}, x^{c/2}y^{\mu_{c/2}}, x^{3c/4}y^{\mu_{3c/4}}, x^c \rangle$. In particular, $\mu(I) = c/4 + 4$ while $\mu(\tilde{I}) = 5$.

Proof. It is clear that $(c/2, \mu_{c/2}) = (c/2, \frac{d}{2}) = (2c/4, 2\mu_{c/4} \bmod d)$ as $2\mu_{c/4} \bmod d \equiv \frac{d}{2}$. Also $(c/2, \mu_{c/2}) = (3c/4 \bmod c, 2\mu_{3c/4})$. Hence $(c/2, \mu_{c/2}) \in I_S \cap I_T$. In particular, $I_S = I_T = \{(0, d), (c/4, \mu_{c/4}), (c/2, \mu_{c/2}), (3c/4, \mu_{3c/4}), (c, 0)\}$ noting $J \subseteq \langle x^{c/2}y^{\mu_{c/2}} \rangle$. ■

Figure 5. The circles (white or black) represent the generators of \tilde{I} and the black points (circles of squares) represent the generators of the ideal I .

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